

# Energy distribution in the dyadosphere of a charged black hole

S. S. Xulu

Department Computer Science  
University of Zululand  
Private Bag X1001  
3886 Kwa-Dlangezwa  
South Africa

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## Abstract

The event horizon of a charged black hole is, according to Ruffini[1] and Preparata *et al.*[2], surrounded by a special region called the *dyadosphere* where the electromagnetic field exceeds the Euler-Heisenberg critical value for electron-positron pair production. We obtain the energy distribution in the dyadosphere region for a Reissner-Nordström black hole. We find that the energy-momentum prescriptions of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg give the same and acceptable energy distribution.

## 1 Introduction

Ruffini[1] introduced a new concept of *dyadosphere* of an electromagnetic black hole to explain gamma ray bursts. He defined the dyadosphere as the region just outside the horizon of a charged black hole whose electromagnetic field strength is larger than the well-known Heisenberg-Euler critical value

$$\mathcal{E}_{crit} = \frac{m_e^2 c^3}{e \hbar} \quad (1)$$

for electron-positron pair production ( $m_e$  and  $e$  respectively denote mass and charge of an electron.) For a Reissner-Nordström space-time, the dyadosphere is described by the radial interval  $r_+ \leq r \leq r_{ds}$  where the horizon

$$r_+ = \frac{GM}{c^2} \left( 1 + \sqrt{1 - \frac{q^2}{GM^2}} \right), \quad (2)$$

forms the inner radius of the dyadosphere, while its outer radius is given by

$$r_{ds} = \sqrt{\left( \frac{\hbar}{m_e c} \right) \left( \frac{GM}{c^2} \right) \left( \frac{m_p}{m_e} \right) \left( \frac{e}{q_p} \right) \left( \frac{Q}{\sqrt{GM}} \right)}, \quad (3)$$

where  $M$  and  $Q$  are mass and charge parameters,  $m_p = \sqrt{\hbar c/G}$  and  $q_c = \sqrt{\hbar c}$  are respectively the Planck mass and Planck charge (see in [1]). Ruffini[1], and Preparata *et al.*[2] have investigated certain properties of the dyadosphere corresponding to the Reissner-Nordström space-time (see also Ruffini *et al.*[3]). They[3] found that the electron-positron pair creation process occur over the entire dyadosphere, excluding the horizon, where the electromagnetic field is Heisenberg-Euler overcritical. The total energy of electron-positron pairs converted from static electric energy and deposited within the dyadosphere is calculated[3] to be:

$$E_{dya} = \frac{1}{2} \frac{Q^2}{r_+} \left(1 - \frac{r_+}{r_{ds}}\right) \left[1 - \left(\frac{r_+}{r_{ds}}\right)^2\right]. \quad (4)$$

Several studies show that, in the presence of a strong electromagnetic field, the velocity of light propagation depends on vacuum polarization states (Birula and Birula[4], Adler[5], De Lorenci *et al.*[6]). Drummond and Hathrell[7] showed that the effect of vacuum polarization may lead to superluminal photon propagation. Daniels and Shore[8] investigated photon propagation around a charged black hole. Their results[8] show that the effect of one-loop vacuum polarization on photon propagation in Reissner-Nordström space-time makes superluminal photon propagation possible. Vacuum polarisation effects thus violate the Principle of Equivalence in interacting field theories. It is therefore of interest to examine further properties of the region where the electromagnetic field exceeds the Euler-Heisenberg critical limit. In this paper we investigate the energy distribution in the dyadosphere of a Reissner-Nordström space-time using the energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg.

Einstein's formulation of energy-momentum covariant conservation laws  $\nabla_a T^{ab} = 0$  ( $T^{ab}$  is the energy-momentum tensor of matter and all nongravitational fields) in the form of Poynting theorem ( $\partial_a \Theta_b^a = 0$ , where  $\Theta_b^a$  is known as the Einstein energy-momentum complex) to include contributions from gravitational field involved the introduction of a pseudotensorial quantity. Owing to the fact that  $\Theta_{ab}$  is not a true tensor (although covariant under linear transformations), Levi-Civita, Schrödinger, and Bauer expressed some doubts at the importance of Einstein's local energy-momentum conservation laws (see in Cattani and De Maria [9]). Einstein showed that his energy-momentum complex provides satisfactory expression for the total energy and momentum of isolated systems. This was followed by many prescriptions: e.g. Landau and Lifshitz, Papapetrou, Weinberg, and many others (see in Refs. [10, 11]). Most of these prescriptions are coordinate dependent and others are not. The physical meaning of these was questioned, and the large number of energy-momentum prescriptions not only fuelled scepticism that different energy-momentum definitions could give unacceptable different energy distributions for a given space-time, but also lead to diverse viewpoints on the possibility of localization of energy-momentum. In a series of papers, Cooperstock[12, 13, 14] hypothesized that in general relativity energy and momentum are located only to the regions of nonvanishing energy-

momentum tensor and that consequently the gravitational waves are not carriers of energy and momentum. Although recent results of Xulu[15] and Bringley[16] support this hypothesis, further investigation of this hypothesis is still required.

The main weaknesses of energy-momentum complexes is that most of these restrict one to make calculations in “Cartesian coordinates”. The alternative concept of quasi-local mass is more attractive because these are not restricted to the use of any special coordinate system. There is also a large number of definitions of quasi-local masses. It has been shown[17] that for a given space-time, many quasi-local mass definitions do not give agreed results. On the other hand, the pioneering contributions of Virbhadra and co-workers ([10, 18, 19]) encouraged numerous researchers (see [16, 20, 21] and references therein) to show that for many space-times several energy-momentum complexes give the same and acceptable energy-momentum distribution for a given space-time. Inspired by Virbhadra’s result, Chang, Nester and Chen[22] demonstrated that by associating each of the energy-momentum complexes of Einstein, Landau and Lifshitz, Møller, Papapetrou, and Weinberg with a legitimate Hamiltonian boundary term, each of these complexes may be said to be quasi-local. Quasi-local energy-momentum are obtainable from a Hamiltonian. This important paper of Chang, *et al*[22] dispels the doubts about the physical meaning of these energy-momentum complexes. Hence, below we use energy-momentum complexes to obtain energy distribution in the dyadosphere of a Reissner-Nordström space-time. In the rest of this paper we use  $G = 1, c = 1$  units and follow the convention that Latin indices take values from 0 to 3 and Greek indices take values from 1 to 3.

## 2 The De Lorenci *et al.* Metric

De Lorenci, Figueiredo, Fliche and Novello [23] calculated the correction for the Reissner-Nordström metric from the first contribution of the Euler-Heisenberg Lagrangian and obtained the following metric

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (5)$$

By making  $\sigma = 0$  we again obtain the Reissner-Nordström metric. De Lorenci *et al.*[23] showed that the correction term  $\frac{\sigma Q^4}{5r^6}$  is of the same order of magnitude as the Reissner-Nordström charge term  $\frac{Q^2}{2r^2}$ .

In order to compute the energy distribution using the energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg we are restricted to the use of “Cartesian coordinates”. Therefore we can express the

above metric (5) in  $T, x, y, z$  coordinates. The coordinates transformation is:

$$\begin{aligned} T &= t + r - \int \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\sigma Q^4}{5r^6} \right)^{-1} dr, \\ x &= r \sin \theta \cos \phi, \\ y &= r \sin \theta \sin \phi, \\ z &= r \cos \theta. \end{aligned} \quad (6)$$

Thus one has the metric in  $T, x, y, z$  coordinates:

$$\begin{aligned} ds^2 &= dT^2 - dx^2 - dy^2 - dz^2 - \\ &\quad \left( \frac{2M}{r} - \frac{Q^2}{r^2} + \frac{\sigma Q^4}{5r^6} \right) \times \left[ dT - \frac{xdx + ydy + zdz}{r} \right]^2. \end{aligned} \quad (7)$$

### 3 Einstein energy-momentum complex

The Einstein energy-momentum complex is given as

$$\Theta_i{}^k = \frac{1}{16\pi} H_i{}^{kl}{}_{,l}, \quad (8)$$

where

$$H_i{}^{kl} = -H_i{}^{lk} = \frac{g_{in}}{\sqrt{-g}} \left[ -g (g^{kn} g^{lm} - g^{ln} g^{km}) \right]_{,m}. \quad (9)$$

$\Theta_0{}^0$  and  $\Theta_\alpha{}^0$  denote for the energy and momentum density components, respectively. (Virbhadra[24] mentioned that though the energy-momentum complex found by Tolman differs in form from the Einstein energy-momentum complex, both are equivalent in import.) The energy-momentum components are expressed by

$$P_i = \int \int \int \Theta_i{}^0 dx^1 dx^2 dx^3. \quad (10)$$

Further Gauss's theorem furnishes

$$P_i = \frac{1}{16\pi} \int \int H_i{}^{0\alpha} \mu_\alpha dS, \quad (11)$$

where  $\mu_\alpha$  is the outward unit normal vector over the infinitesimal surface element  $dS$ .  $P_\alpha$  give momentum components  $P_1, P_2, P_3$  and  $P_0$  gives the energy.

The only required components of  $H_i{}^{jk}$  in the calculation of energy are the following:

$$\begin{aligned} H_0{}^{01} &= \gamma x, \\ H_0{}^{02} &= \gamma y, \\ H_0{}^{03} &= \gamma z, \end{aligned} \quad (12)$$

where

$$\gamma = \frac{4M}{r^3} - \frac{2Q^2}{r^4} + \frac{2Q^4\sigma}{5r^8}. \quad (13)$$

For a surface given by parametric equations  $x = r \sin \theta \cos \phi$ ,  $y = r \sin \theta \sin \phi$ ,  $z = r \cos \theta$  (where  $r$  is constant) one has  $\mu_\beta = \{x/r, y/r, z/r\}$  and  $dS = r^2 \sin \theta d\theta d\phi$ . Now using (12) in (11) over a surface  $r = \text{const.}$ , we obtain:

$$E_{\text{Einst}} = M - \frac{Q^2}{2r} + \frac{\sigma Q^4}{10r^5} \quad (14)$$

In the next Section we obtain the energy distribution for the same metric in Landau and Lifshitz formulation.

## 4 Energy distribution in Landau and Lifshitz Formulation

The symmetric energy-momentum complex of Landau and Lifshitz[25] may be written as

$$L^{ij} = \frac{1}{16\pi} \ell^{ikjl}{}_{,kl} \quad (15)$$

where

$$\ell^{ikjl} = -g(g^{ij}g^{kl} - g^{il}g^{kj}). \quad (16)$$

$L^{00}$  is the energy density and  $L^{0\alpha}$  are the momentum (energy current) density components.  $\ell^{mjnk}$  has symmetries of the Riemann curvature tensor. The expression

$$P^i = \int \int \int L^{i0} dx^1 dx^2 dx^3 \quad (17)$$

gives the energy  $P^0$  and momentum  $P^\alpha$  components. Thus after applying the Gauss theorem, the energy expression is given by

$$E_{LL} = \frac{1}{16\pi} \int \int \ell^{0\alpha 0\beta}{}_{,\alpha} \mu_\beta dS, \quad (18)$$

where  $\mu_\beta$  is the outward unit normal vector over an infinitesimal surface element  $dS$ . In order to calculate the energy component for the De Lorenci *et al.* metric expressed by the line element (7), we need the following non-zero components of  $\ell^{ikjl}$

$$\begin{aligned} \ell^{0101} &= -1 + \left(-1 + \frac{x^2}{r^2}\right) \gamma, \\ \ell^{0102} &= \frac{xy\gamma}{r^2}, \\ \ell^{0103} &= \frac{xz\gamma}{r^2}, \\ \ell^{0202} &= -1 + \left(-1 + \frac{y^2}{r^2}\right) \gamma, \\ \ell^{0203} &= \frac{yz\gamma}{r^2}, \\ \ell^{0303} &= -1 + \left(-1 + \frac{z^2}{r^2}\right) \gamma, \end{aligned} \quad (19)$$

Equation (15) with eqs. (16) and (19) gives the energy density component:

$$L^{00} = \frac{1}{8\pi} \left[ \frac{Q^2}{r^4} - \frac{Q^4 \sigma}{r^8} \right]. \quad (20)$$

Using equations (19) in (18) over a surface  $r = \text{const.}$ , we obtain:

$$E_{\text{LL}} = M - \frac{Q^2}{2r} + \frac{\sigma Q^4}{10r^5} \quad (21)$$

Thus we find the same energy distribution we obtained in the last section. In the next Section we obtain the energy distribution for the same metric in Papapetrou formulation.

## 5 Energy distribution in Papapetrou formulation

The Papapetrou energy-momentum complex[26]:

$$\Omega^{ij} = \frac{1}{16\pi} \mathcal{N}^{ijkl}{}_{,kl} \quad (22)$$

where

$$\mathcal{N}^{ijkl} = \sqrt{-g} (g^{ij}\eta^{kl} - g^{ik}\eta^{jl} + g^{kl}\eta^{ij} - g^{jl}\eta^{ik}) \quad (23)$$

is also symmetric in its indices.  $\Omega^{i0}$  are the energy and momentum density components. Energy and momentum components  $P^i$  are given by

$$P^i = \int \int \int \Omega^{i0} dx^1 dx^2 dx^3. \quad (24)$$

Applying the Gauss theorem, the energy  $E$  for a stationary metric is given by the expression

$$E_P = \frac{1}{16\pi} \int \int \mathcal{N}^{00\alpha\beta}{}_{,\beta} \mu_\alpha dS. \quad (25)$$

To find the energy component of the line element (7), we require the following non-zero components of  $\mathcal{N}^{ijkl}$  :

$$\begin{aligned} \mathcal{N}^{0011} &= -1 - \frac{\gamma}{r} + \frac{\gamma x^2}{r^3}, \\ \mathcal{N}^{0012} &= \frac{\gamma xy}{r^3}, \\ \mathcal{N}^{0013} &= \frac{\gamma xz}{r^3}, \\ \mathcal{N}^{0022} &= -1 - \frac{\gamma}{r} + \frac{\gamma y^2}{r^3}, \\ \mathcal{N}^{0023} &= \frac{\gamma yz}{r^3}, \\ \mathcal{N}^{0033} &= -1 - \frac{\gamma}{r} + \frac{\gamma z^2}{r^3}, \end{aligned} \quad (26)$$

Using the above results in (22) and (23), we obtain

$$\Omega^{00} = \frac{1}{8\pi} \left[ \frac{Q^2}{r^4} - \frac{Q^4\sigma}{r^8} \right] \quad (27)$$

Thus we find the same energy density as we obtained in the last section. We now use Eq. (26) in (25) over a 2-surface (as in the last Section) and obtain

$$E_{\text{Pap}} = M - \frac{Q^2}{2r} + \frac{\sigma Q^4}{10r^5}, \quad (28)$$

which is the same as obtained in the previous section. In the next Section we obtain the energy distribution for the same metric in Weinberg formulation.

## 6 Energy distribution in Weinberg formulation

The symmetric energy-momentum complex of Weinberg [27] is :

$$W^{ik} = \frac{1}{16\pi} \Delta^{ikl}{}_{,l} \quad (29)$$

where

$$\Delta^{ikl} = \frac{\partial h_a^i}{\partial x_i} \eta^{lk} - \frac{\partial h_a^i}{\partial x_l} \eta^{ik} + \frac{\partial h^{al}}{\partial x^a} \eta^{ik} - \frac{\partial h^{ai}}{\partial x^a} \eta^{lk} + \frac{\partial h^{ik}}{\partial x_l} - \frac{\partial h^{lk}}{\partial x_i} \quad (30)$$

and

$$h_{ij} = g_{ij} - \eta_{ij}. \quad (31)$$

$\eta_{ij}$  is the Minkowski metric. The expression

$$P^i = \int \int \int W^{i0} dx^1 dx^2 dx^3. \quad (32)$$

gives the energy  $P^0$  and momentum  $P^\alpha$  components. Once more Gauss's theorem furnishes the following expression for the energy  $E$  of a stationary metric:

$$E_W = \frac{1}{16\pi} \int \int \Delta^{\alpha 0k} \mu_\alpha dS. \quad (33)$$

To find the energy component of the line element (7), we require the following non-zero components of  $\Delta^{ijk}$ :

$$\begin{aligned} \Delta^{100} &= \gamma x, \\ \Delta^{200} &= \gamma y, \\ \Delta^{300} &= \gamma z, \end{aligned} \quad (34)$$

where  $\gamma$  is given by Eq. (13). To find the energy density component  $W^{00}$ , we use Eq. (29) with (30) and (34) and get

$$W^{00} = \frac{1}{8\pi} \left[ \frac{Q^2}{r^4} - \frac{Q^4\sigma}{r^8} \right], \quad (35)$$

which agrees with energy density components of the Landau-Lifshitz and the Papapetrou energy-momentum complexes. Now using (34) in (33) we obtain

$$E_W = M - \frac{Q^2}{2r} + \frac{\sigma Q^4}{10r^5}. \quad (36)$$

which is also the same as in the above sections.

## 7 Conclusion

Misner *et al* [28] argued that to look for a local energy-momentum is looking for the right answer to the wrong question. They further argued that energy is only localizable for spherical systems. Cooperstock and Sarracino [29] countered this point of view, arguing that if energy is localizable in spherical systems then it is localizable in any space-times. Bondi[30] pleaded that a nonlocalizable form of energy is not admissible in general relativity. The viewpoints of Misner *et al* discouraged further study of energy localization and on the other hand an alternative concept of energy, the so-called quasi-local energy, was developed. To date, a large number of definitions of quasi-local mass have been proposed. The uses of quasi-local masses to obtain energy in a curved space-time are not limited to a particular coordinates system whereas many energy-momentum complexes are restricted to the use of “Cartesian coordinates.” Penrose[31] emphasized that quasi-local masses are conceptually very important. Nevertheless, the present quasi-local mass definitions still have inadequacies. For instance, Bergqvist[17] studied seven quasi-local mass definitions and concluded that no two of these definitions give agreed results for the Reissner-Nordström and Kerr space-times. The shortcomings of the seminal quasi-local mass definition of Penrose in handling the Kerr metric are discussed in Bernstein and Tod[32], and in Virbhadra[24]. On the contrary, the remarkable work of Virbhadra, and some others, and recent results of Chang, Nester and Chen have revived the interest in various energy-momentum complexes. Recently Virbhadra stressed that although the energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg are nontensorial (under general coordinate transformations), these do not violate the principle of covariance as the equations describing the conservation laws with these objects are true in any coordinates systems.

In this paper we obtained the energy distribution in De Lorenci *et al.* space-time using the energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou, and Weinberg. All four prescriptions give the same distribution of energy ( $E_{Einst} = E_{LL} = E_{Pap} = E_W$ ) given as:

$$E = M - \frac{Q^2}{2r} + \frac{\sigma Q^4}{10r^5}. \quad (37)$$

It is obvious that in the dyadospheric region (where  $r$  is small) the last term plays a very important role. As expected,  $\sigma = 0$  gives the energy distribution for the Reissner-Nordström metric.



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